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# Comparison of Localized Radial Basis Functions' (LRBF) Solution of the Two-Dimensional Advection–Diffusion Equation to the Finite Difference Methods' (FDM)

*S. M. Mousavi, D. L. Young, C. W. Tsai*
Department of Civil Engineering and Hydrotech Institute, National Taiwan University
Taipei, Taiwan

ABSTRACT

The advection–diffusion (A–D) equation plays an important role in the simulation of suspended sediment transport, contaminant transport and water quality in rivers. In this study, two classes of numerical methods are proposed to solve the two-dimensional advection-diffusion equation. One is the method of approximate particular solutions (MAPS), which is a representative of the meshless numerical family; and the other is the FDM method, a mesh-dependent scheme. We utilize the localization technique (LMAPS) to resolve two well-acknowledged difficulties of the Radial Basis Functions (RBFs). Firstly, the LMAPS is used to overcome the ill-conditioned matrix problem, and secondly the localization technique can assist in making the MAPS less sensitive to the shape parameter. To the best knowledge of the writers, there are no systematic techniques or algorithms to evaluate the accuracy or stability of the localized RBF methods. To evaluate and compare the accuracy and behavior of the LMAPS as a meshless method, a mesh-dependent technique, say FDM, is employed in this study. The proposed numerical schemes are validated against available analytical solutions for two-dimensional transport problem. Compared to the proposed FDM, the LMAPS has satisfactory accuracy and stability. The potential usefulness of the meshless methods for transport problems can be demonstrated in this study.

KEY WORDS: suspended sediment transport; advection–diffusion equation; meshless methods; localization; mesh-dependent methods

INTRODUCTION

Understanding the transport of sediment particles is of fundamental and practical importance to hydraulic engineering. Accurate simulation of suspended sediment transport is essential for water quality management, environmental impact assessment and design of hydraulic structures. Among others, the advection–diffusion (A–D) equation is crucial to the simulation of suspended sediment transport, solute contaminant transport and water quality in rivers. Therefore, improving the efficiency and accuracy of numerical schemes for the A–D equation has been a focus of research (Man and Tsai, 2008).

## Governing equations

As it was mentioned before, the 2D A-D equation is solved in this study:

$$\frac{\partial C}{\partial t} + (\vec{v}.\nabla)C = k\nabla^2 C \quad in \quad \Omega \tag{1}$$

where $C$ is the scalar function which may be a suspended load or pollutant concentration; $k$ is the diffusivity; $t$ is the time; $\vec{v}$ is the given velocity; and $\Omega$ is the computational domain with a boundary $\Gamma$. The solution of the above governing equation has to be obtained for the following initial and boundary data:

Initial conditions (I.C.):

$$C(\vec{x}, t_0) = C_0 \quad \vec{x} \in \Omega \tag{2}$$

Dirichlet and Neumann boundary conditions (B.C.):

$$C(\vec{x}, t) = C_\Gamma \quad \vec{x} \in \Gamma^D \tag{3}$$

$$\frac{\partial C(\vec{x}, t)}{\partial n_\Gamma} = q_\Gamma \quad \vec{x} \in \Gamma^N \tag{4}$$

in which $\Gamma^D$ and $\Gamma^N$ represent the Dirichlet and Neumann part of the boundary $\Gamma$; $\vec{x}$ is the vector space; $t_0$ is the initial time and $n_\Gamma$ is the normal vector on the boundary $\Gamma^N$ and $C_0$; $C_\Gamma$ as well as $q_\Gamma$ are known functions.

## Numerical method

***Euler method.*** The Euler method is a well-known explicit scheme which is based on the first two terms in the expansion of the Taylor series:

$$y_{n+1} = y_n + f(y_n, t_n) \times \Delta t \tag{5}$$

Applying a stability analysis for the Euler method, it can be shown that the step size should be limited to:

$$\Delta t \le \frac{2}{|\lambda|} \tag{6}$$

where $\lambda$ , in this case, is a real, negative constant:

$$\lambda = \frac{\partial f}{\partial y}(y_0, t_0) \tag{7}$$

***Runge-Kutta methods.*** Runge–Kutta (RK) methods introduce points between $t_n$ and $t_{n+1}$ and valuate $f$ at these intermediate points. The additional function evaluations, of course, result in higher cost per time step; but the accuracy is increased. As it turns out, better stability properties are also obtained. With three out of the four constants chosen, we have a one-parameter family of second-order Runge–Kutta (RK2) formulas (Moin, 2010):

$$k_1 = f(y_n, t_n) \times \Delta t \tag{8}$$

$$k_2 = f(y_n + \alpha k_1, t_n + \alpha \Delta t) \times \Delta t \tag{9}$$

$$y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} k_2 \tag{10}$$

In this study, we choose $\alpha = 0.5$ . Again, by applying a stability analysis for RK2, the following criteria can be obtained:

$$|\sigma| = \sqrt{1 + \frac{\omega^4 (\Delta t)^4}{4}} > 1 \tag{11}$$

where $\omega = \lambda / i$ ; and $\lambda$ is the same as in the Eq. )7). The method is unconditionally unstable for purely imaginary $\lambda$ . However, noting that for small values of $\omega \times \Delta t$ , this method is less unstable than the explicit Euler.

***Mesh discretization in FDM.*** Since FDM is a well-known method and to be concise, here the discretized form of the G.E. is provided without extra details:

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \tag{12}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + O(\Delta x^2) \tag{13}$$

For the both first-order and second order derivatives in $y$ direction, a similar expression can be obtained.

***RBFs.*** RBFs were originally introduced in the early 1970s to obtain multivariate scattered data approximations and function interpolations. Notably, in contrast to the traditional meshed-based methods such as finite difference, finite element, and boundary element methods, the RBF collocation methods are mathematically simple and truly meshless, which avoid troublesome mesh generation for high-dimensional problems involving irregular or moving boundary (Chen et al. 2013). Method of approximate particular solution (MAPS) is one of the RBFs which would be discussed in the following.

***The localized method of approximate particular solutions.*** The LMAPS is a developing numerical method that is based on the MAPS. The MAPS is a series of meshless numerical method for solving differential equations that includes various radial basis function collocation methods (RBFCM). Various basis functions such as radial basis functions or polynomial equations can be applied. The MAPS, first was developed by Chen et al. (2011, 2012). This method can be described as the following. A solution of the following form can be found for a given variable $q$ at any computation point within the computational domain $\Omega$ :

$$q(\vec{x}_i) = \sum_{j=1}^{N} \alpha_j F\left(\left\|\vec{x}_i - \vec{x}_j\right\|\right), \ \vec{x} \in \Omega \tag{14}$$

where $F$ is the integrated basis function; and $\alpha$ is the weighting coefficient to be determined. For numerical implementation of any given function or operator $\zeta\{\ \}$ , the MAPS can generate an approximate summation equation for each computation point as:

$$\zeta\{q(\vec{x}_i)\} = \sum_{j=1}^{N} \alpha_j \zeta\left\{F\left(\left\|\vec{x}_i - \vec{x}_j\right\|\right)\right\}, \ \vec{x} \in \Omega \tag{15}$$

By solving the (15, the weighting coefficient of each global point $\vec{x}_i$ is found, and therefore the variable $q$ at any point within the computational domain can be approximated as

$$q(\vec{x}_i) \cong \sum_{j=1}^{N} \alpha_j F\left(\left\|\vec{x}_i - \vec{x}_j\right\|\right), \ \vec{x} \in \Omega \tag{16}$$

## Localization technique

The process of finding the weighting coefficient of each global point requires solving a linear system with full matrix. Full matrix is more likely to cause ill-condition than sparse matrix. The large memory loading and long computational time will hinder the application for large scale computation. In order to overcome the aforementioned problems, localization technique (Yao et al. 2011) is applied to modify the MAPS. For each local influence area $\omega_i$ , the approximation summation equation for variable $q$ can be depicted as:

$$q(\vec{x}) \cong q(\vec{x}_{i,k}) = \sum_{m=1}^{NL} \alpha_{i,m} F\left(\left\|\vec{x}_{i,k} - \vec{x}_{i,m}\right\|\right), \ \vec{x} \in \omega_i \tag{17}$$

According to the collocation method, each point within local influence area should satisfy Eq. )17:

$$\vec{q}_i = \vec{F}_i \vec{\alpha}_i \tag{18}$$

where

$\vec{q}_i = \left[ q\left(\vec{x}_{i,1}, t\right), \ldots, q\left(\vec{x}_{i,NL}, t\right) \right]^T$, $\vec{F}_i = \left[ F\left( \left\| \vec{x}_{i,k} - \vec{x}_{i,m} \right\| \right) \right]_{NL \times NL}$, and $\vec{\alpha}_i = \left[ \alpha_{i,1}, \ldots, \alpha_{i,NL} \right]^T$. Since $\vec{F}_i$ is invertible, the weighting coefficient can be obtained as

$$\vec{\alpha}_i = \vec{F}_i^{-1} \vec{q}_i \tag{19}$$

For numerical implementation of any given function or operator $\zeta\{\ \}$, Eq. )17 can be derived as

$$\zeta\left\{ q\left(\vec{x}\right) \right\} \cong \zeta\left( q\left(\vec{x}_i\right) \right) = \sum_{m=1}^{NL} \alpha_{i,m} \zeta\left\{ F\left( \left\| \vec{x}_i - \vec{x}_{i,m} \right\| \right) \right\} \tag{20}$$

Defining $f = \zeta\{F\}$, we obtain

$$\zeta\left\{ q\left(\vec{x}_i\right) \right\} = \sum_{m=1}^{NL} \alpha_{i,m} f\left( \left\| \vec{x}_i - \vec{x}_{i,m} \right\| \right) \tag{21}$$

Or in vector-matrix form,

$$\zeta\left\{ q\left(\vec{x}_i\right) \right\} = \vec{f}_i \vec{\alpha}_i \tag{22}$$

where $\vec{f}_i = \left[ f\left( \left\| \vec{x}_i - \vec{x}_{i,1} \right\| \right), \ldots, f\left( \left\| \vec{x}_i - \vec{x}_{i,NL} \right\| \right) \right]$. By substituting Eq. )19 into Eq. (22, we have

$$\zeta\left\{ q\left(\vec{x}_i\right) \right\} = \vec{f}_i \vec{F}_i^{-1} \vec{\alpha}_i \tag{23}$$

Let $\vec{\varphi}_i = \vec{f}_i \vec{F}_i^{-1} = \left[ \varphi_{i,1}, \ldots, \varphi_{i,NL} \right]$, the equation for variable $q$ within local influence area can be concluded as

$$\zeta\left\{ q\left(\vec{x}_i\right) \right\} = \vec{\varphi}_i \vec{q}_i \tag{24}$$

Such local system can be applied for local interpolation. However, for boundary value problems, the solution for a local system needs to be expanded to a global system. The transformation from local influence area to global point gives:

$$\vec{\Phi}_i = \left[ \Phi_{i,1}, \ldots, \Phi_{i,N} \right],\ \Phi_{i,j} = \begin{cases} 0, & \vec{x}_j \notin \omega_i \\ \varphi_{i,k}, & \vec{x}_j = \vec{x}_{i,k} \in \omega_i \end{cases} \tag{25}$$

and the global system can be written as

$$\zeta\{\vec{q}\} = \vec{\Phi} \vec{q} \tag{26}$$

Given $\vec{\Phi} = \left[ \vec{\Phi}_1, \ldots, \vec{\Phi}_N \right]^T$, and $\vec{q} = \left[ q\left(\vec{x}_1\right), \ldots, q\left(\vec{x}_N\right) \right]^T$. Finally the solution of $q$ can be obtained directly by solving the linear system of Eq. (26 ( Lin et al. 2015).

There are some relating topics such as local influence area, and normalization techniques which are out of scope of this paper.

## Results and discussions

In this section we will compare the numerical results[1] obtained by the LMAPS as a meshless scheme to the FDM ones as a well-known mesh-dependent method. In this study, I.C. and B.C. are given as following:

I.C.:

$$C\left(x, y, t=0\right) = \sin\left(\pi x\right) + \sin\left(\pi y\right) \tag{27}$$

B.C.:

$$\begin{cases} C\left(x=0, y, t\right) = \left(\sin\left(\pi y\right)\right) e^{-k\pi^2 t} \\ C\left(x=1, y, t\right) = \left(\sin\left(\pi y\right)\right) e^{-k\pi^2 t} \\ C\left(x, y=0, t\right) = \left(\sin\left(\pi x\right)\right) e^{-k\pi^2 t} \\ C\left(x, y=1, t\right) = \left(\sin\left(\pi x\right)\right) e^{-k\pi^2 t} \end{cases} \tag{28}$$

[1] In all cases, diffusivity is 100.

By defining the velocity vector as $\vec{v}=(u,v)=(\cos(\pi y),-\cos(\pi x))$, the analytical solution of the G.E. (1 for the present case can be derived from

$$C(x,y,t)=(\sin(\pi x)+\sin(\pi y))e^{-k\pi^2 t} \tag{29}$$

Since we have the exact solution, Eq. (29, a root-mean-square error (RMSE) parameter is defined as following

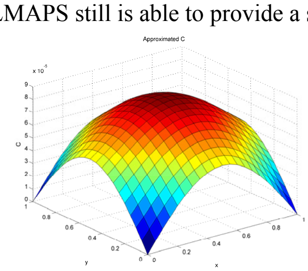
$$RMSE=\sqrt{\frac{\sum_{i=1}^{N}\left(f_{apx_i}-f_{ext_i}\right)^2}{N}} \tag{30}$$

where $N$ is number of total points, $f_{apx_i}$ and $f_{ext_i}$ are the numerical and exact values of the desired function in every points.

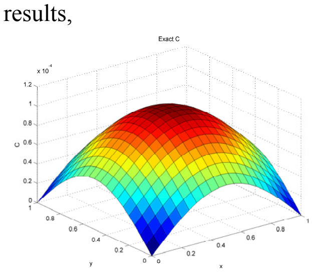
At this juncture, we are ready to present and discuss the results. Fig. 1 shows the results of the LMAPS comparing to the exact solution. In this figure, Euler method has been employed for the transient terms. Schematic results, show a good behavior of the LMAPS, as well as the RMSE (1.6190e-07) is acceptable.

By replacing the FDM method with the LMAPS to the latter example (the Euler scheme for the transient terms and the same time steps within the same size of the domain stencils which were used in the LMAPS method), a good approximation with RMSE = 4.5129e-06 has been observed in Fig. 2. Comparing the RMSE of the LMAPS with the FDM one (almost 28 times lesser), it is clear to see that the LMAPS performs more accurately than the FDM.

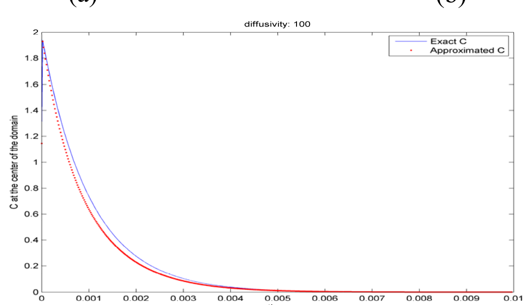
Accuracy is one of the pedagogical aspects of the numerical techniques; sometimes, however, the stability causes lots of difficulties and problems in a numerical solution. By increasing the time steps to $\Delta t=2.0e-05$, which is 20 times bigger than the previous one, the LMAPS still is able to provide a stable results,




(a)



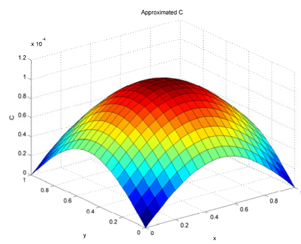

(b)



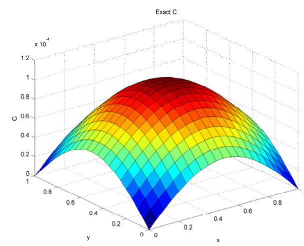

(c)

Fig. 3, as well as it is well accurate (RMSE= 9.1819e-06), while the proposed FDM method failed to converge to a result due to the instability conditions. An interesting point here is that, the LMAPS with $\Delta t=2.0e-05$ provides accurate results close to the proposed FDM with $\Delta t=1.0e-06$.

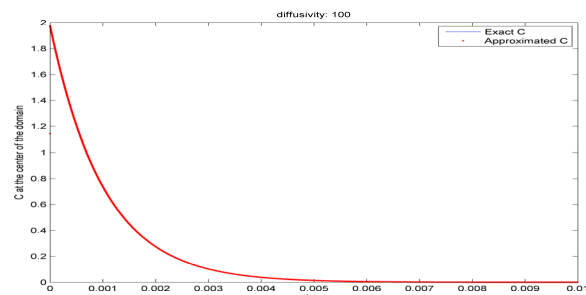
In continuity, we solved the A-D equation with the RK2 method ($\Delta t=1.0e-06$), which is a higher order accurate and more stable method comparing to the Euler scheme, along with the proposed FDM scheme, Fig. 4. The RMSE in this case is 1.6649e-07 which is bigger than its corresponding value in the LMAPS with the Euler scheme (1.6190e-07). In other words, the LMAPS with the Euler scheme is more accurate than the proposed FDM with the RK2.




(a)

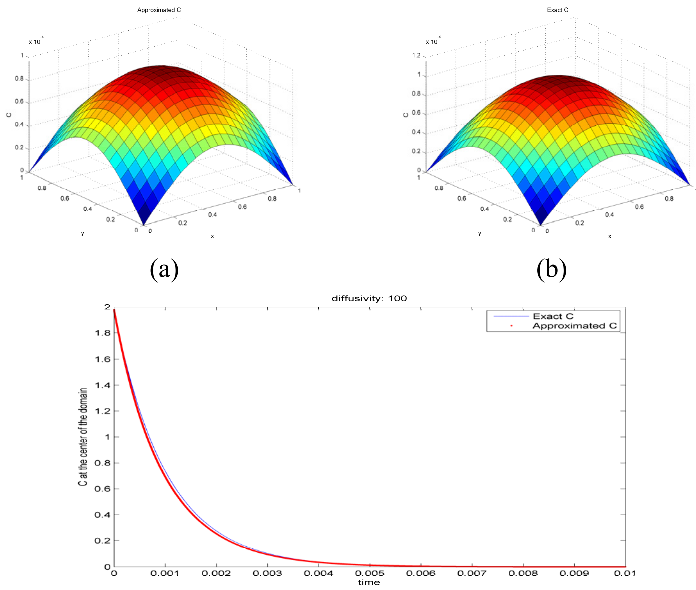



(b)



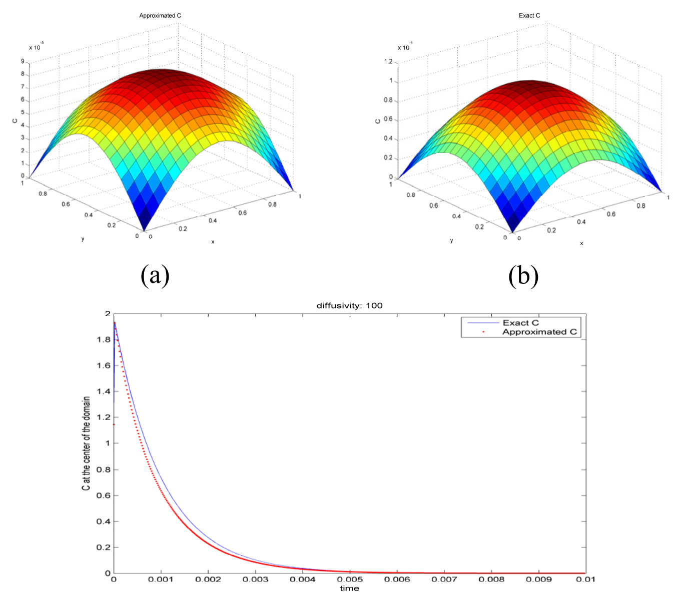

(c)

Fig. 1 (a) The LMAPS solution of the A-D equation with the Euler scheme comparing to the exact one (b), (c) time history of the $C$ at the center point, $\Delta t=1.0e-06$



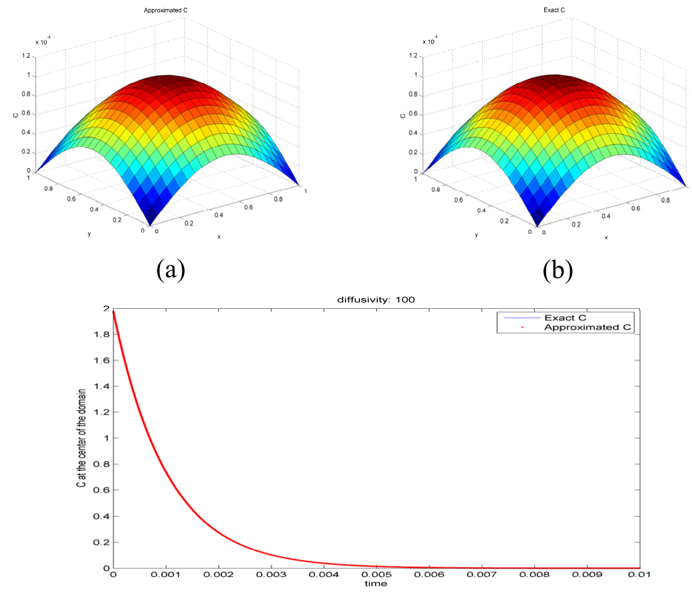

(a) (b) (c)

Fig. 2 (a) The proposed FDM solution of the A-D equation with the Euler scheme comparing to the exact one (b), (c) time history of the $C$ at the center point, $\Delta t = 1.0e-06$




(a) (b) (c)

Fig. 3 (a) The LMAPS solution of the A-D equation with the Euler scheme comparing to the exact one (b), (c) time history of the $C$ at the center point, $\Delta t = 2.0e-05$




(a) (b) (c)

Fig. 4. (a) The proposed FDM solution of the A-D equation with the RK2 scheme comparing to the exact one (b), (c) time history of the $C$ at the center point, $\Delta t = 1.0e-06$

It should be mentioned that despite the LMAPS, the proposed FDM method still could not converge to a solution even with the RK2 scheme for $\Delta t = 2.0e-05$. On a side note, as we have expected here, the RK2 improved the accuracy of the solution which the RMSE (=1.6649e-07) is smaller than the Euler scheme (=4.5129e-06). The accuracy of the mesh-dependent numerical methods can be improved by implementing different spatial discretization methods in addition to changing the temporal scheme. On the other hand, the accuracy of the meshless methods like the LMAPS also can be modified simply by changing the number of local points or the shape parameter[2], while for the FDM, we need to have more points in the discretization of the G.Es, which needs more effort to apply in the numerical coding algorithm. In addition, RBFs are really meshless and much easier to apply in coding especially when it comes to the complicated geometries or higher dimensions. To make a fair judgment, despite the aforementioned advantages for the RBFs, the authors want to mention two drawbacks of this family of numerical methods; firstly, most of the RBFs are based on the shape parameter, which up to now there is no systematic way to obtain the optimum value for it; however, some techniques[3] have been used for finding the optimum shape parameter but they are applicable only for the global RBFs. The second disadvantage of the RBFs is, whether you are using an implicit scheme or not, you always need to solve the inversion of a matrix. Fortunately, with the development of the programing languages, this issue is less considered, since a canned inversion code can be found in almost all the commonly used programing languages such as MATLAB, C family, and Fortran.

## CONCLUSIONS

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[2] The shape parameter is a factor in the F function in Eq. 14.

[3] Like the LOOCV

A 2D A-D equation was solved by two families of the numerical methods: 1- LMAPS, 2- FDM. It was shown that the LMAPS with the Euler scheme, comparing to the proposed FDM with the Euler and the RK2 scheme, provides more stable and accurate results in this case. The potential usefulness of the meshless methods for transport problems was demonstrated in this study.

## REFERENCES

Chen, C. S., Fan, C. M., and Wen, P. H. (2010). The method of approximate particular solutions for solving certain partial differential equations. Numerical Methods for Partial Differential Equations. 28(2): 506-522.

Chen, C. S., Fan, C. M., and Wen, P. H. (2011). The method of approximate particular solutions for solving elliptic problems with variable coefficients. Int. J. Comput. Methods. 8(3): 545-559.

Chen, W., Fu, Z., and Chen, C. (2013). Recent Advances in Radial Basis Function Collocation Methods. New York. Springer press.

Lin, C.Y., Gu, M.H., Young, D.L., Sladek, J., and Sladek, V. (2015). The localized method of approximated particular solutions for solving two-dimensional incompressible viscous flow field. Engineering Analysis with Boundary Elements. 57: 23–36.

Man, C., and Tsai, C. (2008). A higher-order predictor–corrector scheme for two-dimensional advection–diffusion equation. Int. J. Numer. Meth. Fluids. 56:401–418.

Moein, P. (2010) Fundamentals of Engineering Numerical Analysis, New York. Cambridge University Press.

Yao, G., Kolibal, J., and Chen, C. (2011). A localized approach for the method of approximate particular solutions. Computers and Mathematics with Application. 61: 2376-2387.