12th International Conference on Hydroscience & Engineering
Hydro-Science & Engineering for Environmental Resilience
November 6-10, 2016, Tainan, Taiwan.

Towards a High-Order Godunov Method for Dispersive Wave Problems

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ABSTRACT

Our aim in this talk is to describe the recent endeavors towards the development of high-order finite volume methods for the efficient and accurate computation of nonlinear dispersive waves in fluid flows. The Green-Naghdi model for shallow water flow and the Iordanski-Kogarko-Wijngaarden model will be considered as examples in the algorithm for numerical discretization.

KEY WORDS: High-order Godunov method, Green-Naghdi equations, Iordanski-Kogarko-Wijingaarden model

INTRODUCTION

The Green-Naghdi (GN) equations for shallow water flow (cf. [4, 7, 8]) is one of the basic models in fluid flows for the study of dispersive waves and the numerical method development. In one space dimension, the GN model can be written in the form

$$\partial_t h + \partial_x (hu) = 0, \tag{1a}$$

$$\partial_t (hK) + \partial_x \left(hKu + \frac{1}{2} g h^2 \right) = \partial_x \left(\frac{2}{3} h^3 (\partial_x u)^2 \right), \tag{1b}$$

where h denotes the total water depth, u is the average horizontal velocity, and K (to be called the generalized velocity) is defined by

$$K = u - \frac{1}{3h} \partial_x (h^3 \partial_x u). \tag{1c}$$

Note that the homogeneous part of (1) belongs to a class of hyperbolic model, when $h \geq 0$. The inhomogeneous part on the right-hand side of (1), however, contribute to the modelling of dispersive effects of waves in shallow water theory. The coupling between the variables ρ, u and K via (1c) leads to either an elliptic-type problem for u, once h and K are known a priori(e.g., from the solution of (1)), or a numerical differentiation problem for K, when h and u are known (e.g., from the solution reconstruction step). It is easy to see that in the absence of the higher order derivative terms in u, (1) reduces to the classical Saint-Vennant model for shallow water flow [10].

The other example that is of interest here is the Iordanski-Kogarko-Wijngaarden (IKW) model for bubbly flow in liquid (cf. [5, 6, 9]). Here, in one space dimension, we may write the IKW in the form

$$\partial_t \rho + \partial_x (\rho u) = 0, \tag{2a}$$

$$\partial_t (\rho K) + \partial_x (\rho K u + p_g(R)) = \partial_x \left(\frac{a^2 \rho_l}{18 \rho^3 R^4} (\partial_x u)^2 \right), \tag{2b}$$

where the generalized velocity K is defined by

$$K = u - \partial_x \left(\frac{a \rho_l}{3 \rho R} \partial_x u \right). \tag{2c}$$

Here $\rho = \alpha_l \rho_l + \alpha_g \rho_g, u, p_g$, R denote the mixture density, velocity, and gas pressure, and the bubble radius, respectively. The variables α_k and ρ_k are in turn the volume fraction and phasic density of the fluid phase k for $k=l$ (the liquid phase), g (the gas phase); $\alpha_l + \alpha_g = 1$.

It should be mentioned that (2) is derived from the following equation

$$\begin{aligned}
&\partial_t \rho + \partial_x (\rho u) = 0, \\
&\partial_t (\rho u) + \partial_x (\rho u^2 + p) = 0, \\
&\partial_t (\alpha_g \rho_g) + \partial_x (\alpha_g \rho_g u) = 0, \\
&\partial_t N_b + \partial_x (N_b u) = 0
\end{aligned}$$

together with the assumptions:

1. The liquid phase is assumed to be incompressible, where ρ_l is set to be a chosen constant.
2. The volume fraction of the gas is assumed to satisfy the relation

$$\alpha_g = \frac{4}{3} \pi R^3 N_b, \tag{3}$$

where R is the bubble radius.

3. The density of the gas phase ρ_g is assumed to be small, and hence can be neglected, yielding the defnition of the mixture density ρ as

$$\rho = (1 - \alpha_g \rho_g) \rho_t + \alpha_g \rho_g \approx (1 - \alpha_g) \rho_t$$

4. The gas inside the bubble is assumed to be ideal, and be compressed or expanded isentrop- ically and uniformly. The gas pressure, denoted by p_g, can then be written in a function of the bubble radius R as

$$p_g = p_0\left(\frac{R_0}{R}\right)^{3\gamma} \tag{4}$$

Where γ is the ratio of specific heats, and p_0, R_0 are reference pressure and radius, respectively.

5. We assume the mixture pressure p to follow the Rayleigh-Plesset equation (cf. [1]) of the form

$$p = p_g(R) - \rho_l\left[R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2\right] \tag{5}$$

Here $dR/dt = \partial_t R + \vec{u}\cdot\nabla R$ stands for the material derivative of R.

6. The bubble distribution is assumed to be uniform, i.e., the flow is in the absence of bubble breakup and coalescence. In this instance, the mass fraction of each fluid phase, denoted by $Y_k = \alpha_k\rho_k / \rho$ for $k = l, g$, and the number of bubbles per unit mass, denoted by $n_b = N_b/\rho$, would retain their respective initial states for all times (cf. [2, 3]).

With that, the bubble radius R can be expressed as a function of the mixture density ρ as

$$\left(\frac{R}{R_0}\right)^3 = \frac{\rho_0}{\alpha_{g0}}\left(\frac{1}{\rho} - \frac{Y_l}{\rho_l}\right), \tag{6}$$

where z_0 is a reference state for the variable z for $z = R$, ρ, and α_g. For simplicity, (6) is written in the form

$$R = \left(\frac{a}{\rho} - b\right)^{1/3}, \tag{7}$$

where the parameters a and b are problem-dependent constants defined by $a = \rho_0/\alpha_{g0}R_0^3$ and $b = aY_l/\rho_l$, respectively.

The model systems considered here (1) and (2) can be put into the general form as

$$\partial_t q + \partial_x f(q) = \psi(q)$$

We will discuss fractional-step method to find higher order approximation of the solution for this type of model system numerically.

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